

# CURVES ON HEISENBERG INVARIANT QUARTIC SURFACES IN PROJECTIVE 3-SPACE

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**ABSTRACT.** This paper is about the family of smooth quartic surfaces  $X \subset \mathbb{P}^3$  that are invariant under the Heisenberg group  $H_{2,2}$ . For a very general such surface  $X$ , we show that the Picard number of  $X$  is 16 and determine its Picard group. It turns out that the general Heisenberg invariant quartic contains 320 smooth conics and that in the very general case, this collection of conics generates the Picard group.

## 1. INTRODUCTION

Let  $A$  be an Abelian surface over  $\mathbb{C}$ , that is a projective group variety of dimension 2. The subgroup  $A_2 = \{a \in A : 2a = 0\}$  has order 16 and therefore  $A_2 \cong (\mathbb{Z}/2\mathbb{Z})^4$ . The involution  $i : A \rightarrow A : a \mapsto -a$  induces a  $\mathbb{Z}/2\mathbb{Z}$  action on  $A$  and the quotient, which we denote by  $K_A$ , is called the Kummer surface of  $A$ . If  $A$  admits a certain kind of line bundle (see Section 2.1) there is an induced map  $A \rightarrow \mathbb{P}^3$  which factors through an embedding  $K_A \rightarrow \mathbb{P}^3$  such that the image of the Kummer surface is a quartic with 16 nodes. Moreover, the natural action of  $A_2$  on  $K_A$  extends to a linear action on  $\mathbb{P}^3$ . In one set of coordinates  $(x, y, z, w)$  on  $\mathbb{P}^3$  this action is given by identifying  $A_2$  with the subgroup of  $\text{Aut}(\mathbb{P}^3)$  which is generated by the following four transformations:

$$\begin{aligned} (x, y, z, w) &\mapsto (z, w, x, y), & (x, y, z, w) &\mapsto (y, x, w, z) \\ (x, y, z, w) &\mapsto (x, y, -z, -w), & (x, y, z, w) &\mapsto (x, -y, z, -w). \end{aligned}$$

The subject of this paper is the family of all quartic surfaces in  $\mathbb{P}^3$  which are invariant under these transformations. We will refer to such surfaces as *Heisenberg invariant quartics* or just *invariant quartics*. This family of quartics appeared in the classical treatises [19, 23] and also in several later works [4, 30, 37]. The family is parameterized by  $\mathbb{P}^4$  and the subfamily of Kummer surfaces described above constitutes a Zariski open dense subset of a hypersurface  $S_3 \subset \mathbb{P}^4$  known as the Segre cubic. However, the general Heisenberg invariant quartic is smooth. In [4], Barth and Nieto study the locus of Heisenberg invariant quartics that contain a line and find a quintic threefold  $N_5 \subset \mathbb{P}^4$  such that the general point corresponds to a desingularized Kummer surface of an Abelian surface with a  $(1, 3)$  polarization. Prior to that, these surfaces had been discovered by Traynard [36] and discussed by Godeaux [15] and Naruki [31]. The present paper was motivated by the following question: how does the locus of Heisenberg invariant quartics that contain a conic look like? It turns out that a general invariant quartic contains at least 320 smooth conics. The conics are found by a direct computation using the geometry of the family. Another result of this paper is that a very general member of the family

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of surfaces has Picard number 16. This is in accordance with the fact that certain moduli spaces of K3 surfaces whose Picard group contain a fixed lattice  $M$  have dimension  $20 - \text{rank}(M)$ . A Picard group of rank 16 thus fits nicely with a family parameterized by  $\mathbb{P}^4$ . The group action on the surfaces induces an action on the Picard group and we also show that, for a very general surface in the family, the sublattice of invariant divisor classes is generated by the class of the hyperplane section. In particular, any invariant curve on such a surface is a complete intersection. Next we determine the Picard group of a very general Heisenberg invariant quartic and show that it is generated by the 320 smooth conics. This is done by computing the configuration of the 320 conics as well as using some general facts on the existence of curves on Kummer surfaces.

The paper is organized as follows. Section 2 sets the notation and reviews the results that we need for the sequel. In Section 3 we determine the Picard number of a very generic invariant quartic. Section 4 concerns the existence of the 320 conics. In Section 5 and Section 6 we study configurations of lines on Heisenberg invariant quartic surfaces and use the results to determine the configuration of the 320 smooth conics on a very general surface. In Section 7 we determine the Picard group of a very general surface in the family up to isomorphism and show that it is generated by the 320 smooth conics.

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## 2. THE FAMILY OF INVARIANT QUARTICS

**2.1. Kummer surfaces.** We begin with an overview of Kummer surfaces, for proofs and notation see [7, 17, 19, 30]. Let  $A$  be an Abelian surface over  $\mathbb{C}$ . The subgroup of 2-torsion points  $A_2 = \{a \in A : 2a = 0\}$  has order 16 and hence  $A_2 \cong (\mathbb{Z}/2\mathbb{Z})^4$ . To an element  $a \in A$  we associate a translation map  $t_a : A \rightarrow A : x \mapsto x + a$ . The involution  $i : A \rightarrow A : a \mapsto -a$  induces a  $\mathbb{Z}/2\mathbb{Z}$  action on  $A$  and the quotient  $K_A = A/\{1, i\}$  is an algebraic surface called the Kummer surface of  $A$ . Let  $\pi : A \rightarrow K_A$  be the projection. The Kummer surface has 16 singular points, namely  $\pi(A_2)$ , and the action of  $A_2$  on  $A$  by translations induces an action on  $K_A$ . For an ample line bundle  $L$  on  $A$  we define the Heisenberg group

$$H(L) = \{a \in A : t_a^* L \cong L\} \subset A,$$

and the set

$$\mathcal{G}(L) = \{(x, \phi) : x \in H(L), \phi : L \xrightarrow{\cong} t_x^* L\}.$$

If  $(x, \phi) \in \mathcal{G}(L)$  and  $(y, \psi) \in \mathcal{G}(L)$  then there is an induced isomorphism  $t_x^* \psi : t_x^* L \rightarrow t_x^* (t_y^* L)$ . Using that  $t_x^* (t_y^* L) = t_{x+y}^* L$  we put a group structure on  $\mathcal{G}(L)$  by letting

$$(y, \psi)(x, \phi) = (x + y, t_x^* \psi \circ \phi).$$

These two groups are connected by an exact sequence,

$$1 \rightarrow \mathbb{C}^* \rightarrow \mathcal{G}(L) \rightarrow H(L) \rightarrow 1,$$

where  $\mathcal{G}(L) \rightarrow H(L) : (x, \phi) \mapsto x$ . The kernel of the map  $\mathcal{G}(L) \rightarrow H(L)$  is the group of automorphisms of  $L$ , which is the multiplicative group  $\mathbb{C}^*$  acting by multiplication

by constants. We will consider the case where  $A$  has a principal polarization  $L'$ , that is  $L'$  is an ample line bundle on  $A$  whose elementary divisors are both equal to 1. In addition we assume that  $L'$  is symmetric and irreducible. The former means that  $i^*L' \cong L'$  where  $i : A \rightarrow A : a \mapsto -a$ , and the latter means that the polarized Abelian surface  $(A, L')$  does not split as a product of elliptic curves. We say that the line bundle  $L = L' \otimes L'$  is of type  $(2, 2)$ . Then  $H(L) = A_2$ ,  $\dim(\Gamma(A, L)) = 4$  and if  $D$  is a divisor on  $A$  that corresponds to  $L$ , then  $D^2 = 8$ . The group  $\mathcal{G}(L)$  has an action on the space of global sections of  $L$ : for  $z = (x, \phi)$  and  $s \in \Gamma(A, L)$ ,  $\phi$  induces a section  $\phi(s)$  of  $t_x^*L$  and the translation  $t_{-x}^*$  induces a section  $t_{-x}^*(\phi(s))$  of  $L = t_{-x}^*(t_x^*L)$ . Thus we put  $zs = t_{-x}^*(\phi(s))$ . This makes  $\Gamma(A, L)$  into an irreducible  $\mathcal{G}(L)$  module such that  $\mathbb{C}^*$  acts by rescaling. After choosing a basis of  $\Gamma(A, L)$ , this gives a faithful linear action of  $H(L) = A_2$  on  $\mathbb{P}^3$ . The rational map  $A \rightarrow \mathbb{P}^3$  induced by  $L$  is defined everywhere and factors through an embedding  $K_A \rightarrow \mathbb{P}^3$  and the image of the Kummer surface is a quartic with 16 nodes. This is the maximal number of nodes of a quartic surface in  $\mathbb{P}^3$  and any quartic in  $\mathbb{P}^3$  with 16 nodes is a Kummer surface. For each of the nodes  $p$ , there is a plane  $P$  which contains  $p$  and 5 other points in the orbit of  $p$  under  $H(L)$ . Moreover, these 6 points lie on a nondegenerate conic  $C$  and the plane  $P$  touches the Kummer surface along  $C$ . We say that  $P$  is a trope. In total we have 16 tropes arising this way and together with the 16 nodes they form Kummer's  $16_6$  configuration: each plane contains 6 points and there are 6 planes through each point. Now, the  $H(L)$  action on  $\mathbb{P}^3$  restricts to the given  $A_2$  action on  $K_A$  and the embedded Kummer surface is thus invariant under the linear action on the ambient space. The Stone-von-Neumann-Mumford theorem states that, up to isomorphism,  $\mathcal{G}(L)$  has a unique irreducible representation such that  $\mathbb{C}^*$  acts by rescaling. Throughout the paper we will fix coordinates on  $\mathbb{P}^3$  and the particular action of  $H(L)$  given in the introduction. This merely reflects a choice of coordinates though.

**2.2. The Heisenberg group.** Let  $(x, y, z, w)$  be coordinates on  $\mathbb{C}^4$  and consider the following four elements of  $\mathrm{SL}_4(\mathbb{C})$ :

$$\begin{aligned}\sigma_1 &: (x, y, z, w) \mapsto (z, w, x, y) \\ \sigma_2 &: (x, y, z, w) \mapsto (y, x, w, z) \\ \tau_1 &: (x, y, z, w) \mapsto (x, y, -z, -w) \\ \tau_2 &: (x, y, z, w) \mapsto (x, -y, z, -w).\end{aligned}$$

Let  $H_{2,2}$  be the subgroup of  $\mathrm{SL}_4(\mathbb{C})$  generated by  $\sigma_1, \sigma_2, \tau_1$ , and  $\tau_2$ . This group is of order 32 and the center and commutator of  $H_{2,2}$  are both equal to  $\{1, -1\}$ , where 1 denotes the identity matrix of size 4. Let

$$H = H_{2,2}/\{1, -1\} \subset \mathrm{Aut}(\mathbb{P}^3),$$

a group of order 16. Since every element of  $H$  has order 2, it follows that

$$H \cong (\mathbb{Z}/2\mathbb{Z})^4.$$

In the sequel we will refer to  $H$  as the *Heisenberg group*.

We consider  $H$  as an  $\mathbb{F}_2$ -vector space  $(\mathbb{F}_2)^4 \cong H$ . It carries a symplectic form given by  $\langle g, h \rangle = 0$  if  $g, h \in H_{2,2}$  commute and  $\langle g, h \rangle = 1$  if they anticommute. For each element  $g \in H_{2,2}$ ,  $g \neq \pm 1$ , the fixed points of  $g \in H$  form two skew lines in  $\mathbb{P}^3$ . These correspond to the eigenspaces of  $g$  in  $\mathbb{C}^4$ , with eigenvalues  $\pm 1$  or  $\pm\sqrt{-1}$ . In total we have 30 such lines in  $\mathbb{P}^3$  which will be called the *fix lines*. If  $h \in H_{2,2}$  commutes with  $g$ , then  $h$  leaves the fix lines belonging to  $g$  invariant and,

if  $h$  anticommutes with  $g$ , then it flips the fix lines of  $g$ . Two fix lines belonging to  $g, h \in H_{2,2}$  where  $g \neq \pm h$ , intersect if and only if  $g$  and  $h$  commute.

Let  $\mathfrak{S}_n$  denote the symmetric group on  $n$  letters and let  $N$  be the normalizer of  $H_{2,2}$  in  $\mathrm{SL}_4(\mathbb{C})$ ,

$$N = \{n \in \mathrm{SL}_4(\mathbb{C}) : nH_{2,2}n^{-1} = H_{2,2}\}.$$

Let  $n \in N$ . The reason for introducing  $N$  is that if a subset  $X \subseteq \mathbb{P}^3$  is invariant under  $H$  and  $n \in N$ , then  $nX$  is also invariant. In fact, for any  $g \in H_{2,2}$  there is an  $h \in H_{2,2}$  such that  $gn = nh$ , and hence  $g(nX) = n(hX) = nX$ . We now proceed to give a useful relation between  $N$  and  $\mathfrak{S}_6$ . Since  $N$  acts on  $H_{2,2}$  by conjugation, and acts as the identity on the center  $\{1, -1\}$ ,  $N$  acts on the vector space  $H_{2,2}/\{1, -1\} \cong (\mathbb{F}_2)^4$ . This action is linear and the transformations preserve the symplectic form. Thus we have a map

$$\phi : N \rightarrow \mathrm{Sp}_4(\mathbb{F}_2).$$

The kernel obviously contains  $H_{2,2}$ . In fact it contains the group  $\langle \pm i, H_{2,2} \rangle$  of order 64 which is generated by  $H_{2,2}$  and multiplication by  $\pm i$ , where  $i = \sqrt{-1}$ . By [9] Theorem 118,  $\mathrm{Sp}_4(\mathbb{F}_2) \cong \mathfrak{S}_6$  and we get a sequence

$$1 \rightarrow \langle \pm i, H_{2,2} \rangle \rightarrow N \rightarrow \mathfrak{S}_6 \rightarrow 1.$$

In [32] it is shown that this sequence is exact, and thus  $N/\langle \pm i, H_{2,2} \rangle \cong \mathfrak{S}_6$ .

**2.3. The family of surfaces.** Consider the following elements of  $\mathbb{C}[x, y, z, w]$ :

$$\begin{aligned} g_0 &= x^4 + y^4 + z^4 + w^4, & g_1 &= 2(x^2y^2 + z^2w^2), & g_2 &= 2(x^2z^2 + y^2w^2), \\ g_3 &= 2(x^2w^2 + y^2z^2), & g_4 &= 4xyzw. \end{aligned}$$

For  $\lambda = (A, B, C, D, E) \in \mathbb{C}^5$  define

$$(1) \quad F_\lambda = Ag_0 + Bg_1 + Cg_2 + Dg_3 + Eg_4.$$

Then  $\{F_\lambda : \lambda \in \mathbb{C}^5\} \subset \mathbb{C}[x, y, z, w]$  is the set of all homogeneous quartic polynomials in  $\{x, y, z, w\}$  invariant under  $H_{2,2}$ . Now let  $\mathcal{X} \rightarrow \mathbb{P}^4$  be the corresponding family of Heisenberg invariant quartic surfaces:

$$\mathbb{P}^4 \times \mathbb{P}^3 \supset \mathcal{X} = \{(\lambda, p) : F_\lambda(p) = 0\}.$$

There is an linear embedding of  $\mathbb{P}^4$  into  $\mathbb{P}^5$  which better exposes the symmetry of the situation. The normalizer  $N$  of  $H_{2,2}$  in  $\mathrm{SL}_4(\mathbb{C})$  acts on the 5-dimensional vector space of  $H_{2,2}$ -invariant polynomials via the natural action on  $\mathbb{C}^4$ . Following [4, 19, 32] we use six invariant quartic polynomials  $t_0, \dots, t_5$ , which satisfy the relation  $\sum_i t_i = 0$ , to embed the parameter space in  $\mathbb{P}^5$ :

$$\begin{aligned} t_0 &= \frac{1}{3}g_0 - g_1 - g_2 - g_3, & t_1 &= \frac{1}{3}g_0 - g_1 + g_2 + g_3, \\ t_2 &= \frac{1}{3}g_0 + g_1 - g_2 + g_3, & t_3 &= \frac{1}{3}g_0 + g_1 + g_2 - g_3, \\ t_4 &= -\frac{2}{3}g_0 + 2g_4, & t_5 &= -\frac{2}{3}g_0 - 2g_4. \end{aligned}$$

The polynomials  $t_0, t_1, t_2, t_3, t_4, t_5$  generate the space of all  $H_{2,2}$  invariant quartic polynomials. The action of  $N/\langle \pm i, H_{2,2} \rangle \cong \mathfrak{S}_6$  on the polynomial ring permutes these six elements and the group acts on the set  $\{t_0, t_1, t_2, t_3, t_4, t_5\}$  as the full permutation group. We use homogeneous coordinates  $(u_0, u_1, u_2, u_3, u_4, u_5)$  on  $\mathbb{P}^5$ ,

expressing an invariant quartic polynomial as  $\sum_i t_i u_i$ . Let  $U \cong \mathbb{P}^4$  be the linear subspace of  $\mathbb{P}^5$  given by

$$U = \{u_0 + u_1 + u_2 + u_3 + u_4 + u_5 = 0\} \subset \mathbb{P}^5.$$

The parameter space  $\mathbb{P}^4$  of  $\mathcal{X}$  with coordinates  $(A, B, C, D, E)$  is identified with  $U$  via the relations

$$\begin{aligned} A &= -u_4 - u_5, \\ B &= -u_0 - u_1 + u_2 + u_3, \\ C &= -u_0 + u_1 - u_2 + u_3, \\ D &= -u_0 + u_1 + u_2 - u_3, \\ E &= 2u_4 - 2u_5. \end{aligned}$$

The transformation from  $U$  to the parameters  $(A, B, C, D, E)$  may be written as

$$\begin{aligned} u_0 &= A - B - C - D, \\ u_1 &= A - B + C + D, \\ u_2 &= A + B - C + D, \\ u_3 &= A + B + C - D, \\ u_4 &= -2A + E, \\ u_5 &= -2A - E. \end{aligned}$$

The point is that  $\mathfrak{S}_6$  acts on  $U$  by permuting the coordinates, and on the level of the quartic surfaces the action is by means of projective transformations on  $\mathbb{P}^3$ .

Let  $\pi : \mathcal{X} \rightarrow U$  be the projection. For a point  $u \in U$  we have a corresponding surface  $\mathcal{X}_u = \pi^{-1}(u)$ . By *Heisenberg invariant quartic surface* (or simply *invariant quartic*), we shall understand a fiber  $\mathcal{X}_u$  for some  $u \in U$ . In addition to these there are 15 pencils of quartic surfaces in  $\mathbb{P}^3$  which are invariant under  $H$ , but whose defining polynomials are not invariant under  $H_{2,2}$ . These will not be considered, but we mention in passing that the 35 dimensional space of quartic homogeneous polynomials in 4 variables splits in the irreducible  $H_{2,2}$  module spanned by  $g_0, g_1, g_2, g_3, g_4$  and 15 irreducible modules of dimension 2, giving rise to the 15 pencils.

**2.4. The Segre cubic, the Igusa quartic and the Nieto quintic.** In this section we introduce three hypersurfaces in  $\mathbb{P}^4$  relevant for the study of Heisenberg invariant quartics. If a group  $G$  acts on a set  $A$  we use  $\text{orb}_G(a)$  to denote the orbit of a point  $a \in A$ .

First consider the following sets of points, planes and 3-planes in the parameter space  $U$ :

$$\begin{aligned} q_0 &= (1, 1, 1, -1, -1, -1) \\ t_0 &= (1, -1, 0, 0, 0, 0) \\ p_0 &= \{u_0 + u_1 = u_2 + u_3 = u_4 + u_5 = 0\} \\ e_0 &= \{u_0 + u_1 = u_0 + u_1 + u_2 + u_3 + u_4 + u_5 = 0\} \\ \mathcal{Q} &= \text{orb}_{\mathfrak{S}_6}(q_0) \quad (10 \text{ points}) \\ \mathcal{T} &= \text{orb}_{\mathfrak{S}_6}(t_0) \quad (15 \text{ points}) \\ \mathcal{P} &= \text{orb}_{\mathfrak{S}_6}(p_0) \quad (15 \text{ planes}) \\ \mathcal{E} &= \text{orb}_{\mathfrak{S}_6}(e_0) \quad (15 \text{ 3-planes}). \end{aligned}$$

These points, planes and 3-planes relate to a  $\mathfrak{S}_6$ -invariant cubic hypersurface  $S_3 \subset U \cong \mathbb{P}^4$  known as the *Segre cubic*. In  $\mathbb{P}^5$  it is defined by the equations

$$S_3 = \{u_0^3 + u_1^3 + u_2^3 + u_3^3 + u_4^3 + u_5^3 = u_0 + u_1 + u_2 + u_3 + u_4 + u_5 = 0\}.$$

The Segre cubic has ten nodes, namely the  $\mathfrak{S}_6$ -orbit  $\mathcal{Q}$ . It also contains the 15 points of  $\mathcal{T}$ , and the set of 3-planes  $\mathcal{E}$  is the set of tangent spaces to  $S_3$  at these points. For  $t \in \mathcal{T}$  the intersection between  $S_3$  and the tangent space  $T_t S_3$  is a union of three 2-planes. In total we have 15 such planes in  $S_3$ , called the *Segre planes*. This is the set  $\mathcal{P}$ . In [33], Nieto follows Jessop [23] in proving the following statement.

**Proposition 2.1.** *Let  $u = (u_0, \dots, u_5) \in U$ . The surface  $\mathcal{X}_u$  is singular if and only if*

$$(u_0^3 + u_1^3 + u_2^3 + u_3^3 + u_4^3 + u_5^3) \prod_{i < j} (u_i + u_j) = 0.$$

*In other words, the subset of  $U$  parameterizing the singular invariant quartics is the 3-fold*

$$S_3 \cup \bigcup_{t \in \mathcal{T}} T_t S_3.$$

It is shown in [19] and [23] that a general point on the Segre cubic corresponds to an invariant surface with exactly 16 nodes which constitute an orbit under the Heisenberg group. Such a surface acquires Kummer's  $16_6$  configuration of nodes and tropes and is a Kummer surface associated to some Abelian surface.

**Proposition 2.2.** *For generic  $u \in S_3$ ,  $\mathcal{X}_u$  is a Kummer surface.*

For  $u \in U$ , we say that  $\mathcal{X}_u$  is a *quadric of multiplicity two*, if its defining polynomial is a square of a degree two polynomial. By *invariant tetrahedron* we understand a Heisenberg invariant quartic surface which is a union of four planes. One checks by direct computation that there are exactly 10 quadrics of multiplicity two and 15 invariant tetrahedra in the family. These correspond to the 10 nodes  $\mathcal{Q}$  of  $S_3$  and the 15 points  $\mathcal{T} \subset S_3$ . The quadrics of multiplicity two are called the *fundamental quadrics*. There is an explanation of the fundamental quadrics and invariant tetrahedra in terms of the symplectic structure on  $H \cong (\mathbb{F}_2)^4$ . This vector space has 35 planes, each consisting of 4 points. If  $g, h \in H_{2,2}$  span a plane in  $H$ , then that plane will be called isotropic if  $g$  and  $h$  commute and anisotropic if they anticommute. Of the 35 planes, 15 are isotropic and 20 are anisotropic. For an isotropic plane  $\{1, g, h, gh\}$ , the three pairs of fix lines belonging to  $g, h$  and  $gh$  are the edges of an invariant tetrahedron. Similarly the anisotropic planes explain the fundamental quadrics. A plane is isotropic if and only if it is equal to its orthogonal complement and therefore the anisotropic planes form 10 pairs of orthogonal planes. For each such pair  $P_1, P_2$ , the 6 fix lines belonging to elements of  $P_1$  are skew and likewise for  $P_2$ . Moreover, the 6 fix lines belonging to elements of  $P_1$  intersect all the 6 fix lines belonging to elements of  $P_2$ . All 12 lines lie on a fundamental quadric, with 6 lines in one ruling and 6 lines in the other.

The dual variety of  $S_3$  is a quartic hypersurface in  $\mathbb{P}^4$  called the *Igusa quartic*, which we denote by  $I_4$ . The singular locus of  $I_4$  consists of 15 lines. One way to view  $I_4$  is to look at the linear system of invariant quartics. Recall the basis

$\{g_0, \dots, g_4\}$  of this system introduced above. Because the system does not have a base point we have a morphism

$$\alpha : \mathbb{P}^3 \rightarrow \mathbb{P}^4 : q \mapsto (g_0(q), \dots, g_4(q)).$$

The image of  $\alpha$  is  $I_4$  [1]. For a hyperplane  $L \subset \mathbb{P}^4$  we have a corresponding Heisenberg invariant quartic surface  $X_L = \alpha^{-1}(L)$ . We conjecture that, for any hyperplane  $L \subset \mathbb{P}^4$ ,  $L \cap I_4$  is the quotient of  $X_L = \alpha^{-1}(L)$  by the Heisenberg group and  $\alpha : X_L \rightarrow L \cap I_4$  is the quotient map. Regarding the conjecture, see [37] and [1] page 210. Also, note that we at least have that all the points in a given orbit of the Heisenberg group are mapped to the same point by  $\alpha$ .

A generic hyperplane  $L \subset \mathbb{P}^4$  intersects  $I_4$  in a quartic surface with 15 singular points and 10 tropes. The singular points come from the intersections with the singular lines of  $I_4$ . On the quotient  $X_L/H$ , there are 15 singularities that come from points with nontrivial stabilizers in  $H$ . Namely, for any of the 15 nonidentity elements of  $H$ , the two fix lines intersect  $X_L$  in an 8 point orbit that maps to one singular point on  $X_L/H$  by the quotient map. Now, if  $L$  is a generic tangent space to  $I_4$ , then  $I_4 \cap L$  has an additional singularity at the point of tangency and  $I_4 \cap L$  is a Kummer surface. In this way  $I_4$  is a compactification of a moduli space of principally polarized Abelian surfaces with a level-2 structure [14, 19].

Let us now look at one more  $\mathfrak{S}_6$  invariant threefold in the parameter space  $U \cong \mathbb{P}^4$ , the *Nieto quintic*  $N_5$ . This is the Hessian variety to  $S_3$ , that is the zeros of the determinant of the Hessian matrix of the polynomial defining  $S_3$ . The singular locus of  $N_5$  consists of 20 lines together with the 10 nodes of  $S_3$ . With our choice of coordinates, viewing  $\mathbb{P}^4 \subset \mathbb{P}^5$ ,  $N_5$  is defined by the equations

$$\sum_{i=0}^5 u_i = \sum_{i=0}^5 \frac{1}{u_i} = 0,$$

where  $\sum_{i=0}^5 \frac{1}{u_i} = u_1 u_2 u_3 u_4 u_5 + u_0 u_2 u_3 u_4 u_5 + \dots + u_0 u_1 u_2 u_3 u_4$ . The Nieto quintic and its connection to Heisenberg invariant quartics which contain lines is studied in [4] where the following is proved ([4] Sections 7 and 8):

**Proposition 2.3.** *The locus in  $U$  parameterizing Heisenberg invariant quartics which contain a line is equal to  $N_5$  together with the 10 tangent cones to the isolated singular points of  $N_5$  (the 10 nodes of  $S_3$ ).*

### 3. THE PICARD NUMBER

We will show that the Picard number of a Heisenberg invariant quartic  $X$  is 16, if  $X$  is outside a countable union of proper subvarieties. A corollary to this is that any invariant divisor is a multiple of the hyperplane section. We start with some generalities on lattices, Picard groups of K3 surfaces and symplectic group actions on K3 surfaces.

A lattice  $M$  is a finitely generated free Abelian group equipped with an integral symmetric bilinear form. An important invariant of  $M$  is the discriminant, denoted  $\text{discr}(M)$ , which is equal to the determinant of any matrix representing the bilinear form. We shall deal only with non-degenerate lattices, that is we assume that the discriminant is non-zero. If  $N \subseteq M$  is a sublattice of full rank, then  $\text{discr}(N)/\text{discr}(M) = [M : N]^2$ , where  $[M : N]$  is the index of  $N$  in  $M$ , see [3] 1.2.1. If  $N \subseteq M$  is a sublattice, we use the notation  $N^\perp$  for the orthogonal complement of  $N$  in  $M$ . A lattice is called even if the  $\mathbb{Z}$  valued quadratic form

associated to the bilinear form takes on even values only. The signature of a lattice  $M$  is a pair of numbers  $(a, b)$ , where  $a$  is the number of positive eigenvalues of any matrix representing the induced bilinear form on  $M \otimes_{\mathbb{Z}} \mathbb{Q}$  and  $b$  is the number of negative eigenvalues. The sum of  $a$  and  $b$  is the rank of  $M$ .

For a scheme  $X$ , the Picard group is the group of isomorphism classes of line bundles on  $X$ , and is denoted  $\text{Pic}(X)$ . If  $X$  is a smooth variety over an algebraically closed field, then  $\text{Pic}(X)$  is isomorphic to the group of Weil divisors on  $X$  modulo linear equivalence. If  $X$  is a smooth and proper variety over an algebraically closed field, let  $\text{Pic}^0(X)$  denote the subgroup of  $\text{Pic}(X)$  corresponding to divisors algebraically equivalent to 0 and define the Néron-Severi group by  $\text{NS}(X) = \text{Pic}(X)/\text{Pic}^0(X)$ . The Néron-Severi group is a finitely generated Abelian group and its rank is called the Picard number of  $X$ . Now suppose that  $X$  is a projective K3 surface over  $\mathbb{C}$ , that is a smooth simply connected projective surface over  $\mathbb{C}$  with trivial canonical bundle. In this case  $\text{Pic}^0(X) = \{0\}$  and thus  $\text{NS}(X) = \text{Pic}(X)$  and  $\text{Pic}(X)$  is free and finitely generated. It is well known that  $1 \leq \rho(X) \leq 20$ , where  $\rho(X) = \text{rank}(\text{Pic}(X))$  is the Picard number of  $X$ . The second cohomology group  $H^2(X, \mathbb{Z})$  is also free and of rank 22, and the cup product defines a bilinear form giving it the structure of a lattice. Abstractly this lattice is isomorphic to the so called K3 lattice  $L = 3U \oplus 2(-E_8)$ , where  $U$  is the hyperbolic plane, a rank 2 lattice with bilinear form given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and  $E_8$  is the lattice of rank 8 whose bilinear form is given by the Cartan matrix of the root system  $E_8$ . The Picard group of  $X$  may be viewed as a non-degenerate sublattice of  $H^2(X, \mathbb{Z})$  with signature  $(1, \rho(X) - 1)$ , where the bilinear form on  $\text{Pic}(X)$  is the intersection product. Since the lattice  $L$  is even, so is  $\text{Pic}(X)$ . The orthogonal complement of  $\text{Pic}(X)$  in  $H^2(X, \mathbb{Z})$  is called the transcendental lattice and is denoted by  $T_X$ .

If a finite group  $G$  acts on a K3 surface  $X$ , then  $G$  acts on  $H^2(X, \mathbb{Z})$  and the action restricts to the sublattices  $\text{Pic}(X)$  and  $T_X$ . We will write this action  $g^*l$  for  $g \in G$  and  $l \in H^2(X, \mathbb{Z})$ . Let

$$H^2(X, \mathbb{Z})^G = \{l \in H^2(X, \mathbb{Z}) : g^*l = l, \text{ for all } g \in G\},$$

and  $\text{Pic}(X)^G = H^2(X, \mathbb{Z})^G \cap \text{Pic}(X)$ . Since the canonical bundle of a K3 surface  $X$  is trivial,  $X$  has a unique non-zero holomorphic 2-form  $\omega$ , up to rescaling. Hence for  $f \in \text{Aut}(X)$ ,  $f^*\omega = \lambda\omega$ , where  $\lambda \in \mathbb{C}$ . If  $\lambda = 1$ , then  $f$  is called symplectic. In the following proposition, the first claim is due to Nikulin [34] and the second claim is due to Mukai [29].

**Proposition 3.1.** *Let  $G$  be a finite group of symplectic automorphisms of a K3 surface  $X$  and let  $\Omega_G = (H^2(X, \mathbb{Z})^G)^\perp$  be the orthogonal complement of  $H^2(X, \mathbb{Z})^G$  in  $H^2(X, \mathbb{Z})$ . For  $g \in G$ ,  $g \neq 1$ , let  $f(g)$  be the number of fixed points of  $g \in G$  and put  $f(1) = 24$ . Then*

- (1)  $T_X \subseteq H^2(X, \mathbb{Z})^G$  and  $\Omega_G \subseteq \text{Pic}(X)$ ,
- (2)  $\text{rank}(H^2(X, \mathbb{Z})^G) + 2 = \frac{1}{|G|} \sum_{g \in G} f(g)$ .

The generic member of our family is a smooth quartic in  $\mathbb{P}^3$  and hence a K3 surface. Since the action of the Heisenberg group is induced by a representation



$H_{2,2} \rightarrow \mathrm{SL}_4(\mathbb{C})$  and the action of  $H_{2,2}$  preserves the defining polynomials of our surfaces, the action of  $H$  is symplectic by [29] Lemma 2.1.

**Corollary 3.2.** *If  $X \subset \mathbb{P}^3$  is a generic Heisenberg invariant quartic, then*

- (1)  $\mathrm{rank}(H^2(X, \mathbb{Z})^H) = 7$ ,
- (2)  $16 \leq \rho(X)$ .

*Proof.* Let  $g \in H$ ,  $g \neq 1$ . The fixed points of  $g$  form two skew lines in  $\mathbb{P}^3$ , and hence  $g$  has 8 fixed points on  $X$ . By Proposition 3.1,  $T_X \subseteq H^2(X, \mathbb{Z})^H$  and  $\mathrm{rank}(H^2(X, \mathbb{Z})^H) = 7$ . Assume that  $\mathrm{rank}(T_X) = 7$ . The hyperplane class  $h \in \mathrm{Pic}(X)$  is invariant and thus there is an integer  $n > 0$  such that  $nh \in T_X$ . Since  $T_X$  is the orthogonal complement of  $\mathrm{Pic}(X)$ ,  $0 = (nh)h = 4n$ , a contradiction. Hence  $\mathrm{rank}(T_X) \leq 6$ . It follows that  $16 \leq \mathrm{rank}(\mathrm{Pic}(X))$ , since  $\mathrm{rank}(\mathrm{Pic}(X)) + \mathrm{rank}(T_X) = \mathrm{rank}(H^2(X, \mathbb{Z})) = 22$ .  $\square$

For homogeneous  $F \in \mathbb{C}[x, y, z, w]$ , let  $X = \{F = 0\} \subset \mathbb{P}^3$  and consider the Hessian surface of  $X$ ,

$$\mathrm{Hess}(X) = \{x \in \mathbb{P}^3 : \det(\mathrm{Hess}(F)(x)) = 0\},$$

where  $\mathrm{Hess}(F)$  is the Hessian matrix of second derivatives of  $F$ . Let  $T \in \mathrm{GL}_4(\mathbb{C})$  and denote the induced automorphism of  $\mathbb{P}^3$  also by  $T$ . Let  $Y = T(X)$  and let  $(T^{-1})^t$  denote the transpose of  $T^{-1}$ . Note that for  $p \in \mathbb{C}^4$ ,

$$\mathrm{Hess}(F \circ T^{-1})(p) = (T^{-1})^t \cdot \mathrm{Hess}(F)(T^{-1}(p)) \cdot T^{-1}.$$

It follows that  $T$  takes the Hessian surface of  $X$  to the Hessian surface of  $Y$ , with preserved multiplicities of irreducible components. In particular, the Hessian surface of a Heisenberg invariant quartic is a Heisenberg invariant octic.

**Example 3.3.** Let  $u = (1, 1, 1, 1, -2, -2) \in U$  and put  $F = x^4 + y^4 + z^4 + w^4$ . We will look at linear transformations of the Fermat quartic  $X = \mathcal{X}_u = \{(x, y, z, w) \in \mathbb{P}^3 : x^4 + y^4 + z^4 + w^4 = 0\}$ .

There are 15 surfaces in the family that are projectively equivalent to  $X$  corresponding to the  $\mathfrak{S}_6$ -orbit of  $u = (1, 1, 1, 1, -2, -2)$ , and we shall see that there are no others. The Hessian surface of the Fermat quartic is the zeros of  $x^2y^2z^2w^2$ , an invariant tetrahedron of double multiplicity. Suppose that  $T \in \mathrm{GL}_4(\mathbb{C})$  takes  $X$  to some Heisenberg invariant quartic  $Y$ . By the discussion above, the Hessian surface of  $Y$  must be one of the 15 invariant tetrahedra, counted with multiplicity 2. As is outlined below,  $X$  is the only smooth surface in the family which has  $\mathrm{Hess}(X)$  as its Hessian surface. By  $\mathfrak{S}_6$ -invariance, for any  $\sigma \in \mathfrak{S}_6$ ,  $\mathcal{X}_{\sigma u}$  is also determined by its Hessian surface in the sense that it is the only smooth surface in the family with that Hessian. It follows that  $Y$  is in the  $\mathfrak{S}_6$ -orbit of  $X$  and in particular there is only a finite number of Heisenberg invariant quartics that are projectively equivalent to  $X$ .

A Gröbner basis calculation using [18] shows that the system of equations in  $(A, B, C, D, E)$  determined by the condition that the determinant of the Hessian matrix of (1) is proportional to  $x^2y^2z^2w^2$  has exactly 5 solutions in  $\mathbb{P}^4$ , namely the five points given by letting all the coordinates  $(A, B, C, D, E)$  but one be zero. Hence there are exactly five invariant quartics whose Hessian surfaces are equal to  $\mathrm{Hess}(X)$  (as a scheme). The case  $A \neq 0$  is the Fermat quartic itself and the other four surfaces are all reducible: the cases  $B \neq 0$ ,  $C \neq 0$  and  $D \neq 0$  are all unions of two quadrics and the case  $E \neq 0$  is the invariant tetrahedron defined by  $xyzw = 0$ .

A K3 surface  $X$  is called quasi-polarized (pseudo-ample polarized) if there is a line bundle on  $X$  with positive self-intersection that is also numerically effective. Let  $\mathcal{F}_4$  be the coarse moduli space of quasi-polarized K3 surfaces of degree 4 [11, 26]. In particular,  $\mathcal{F}_4$  parameterizes K3 surfaces that embed into  $\mathbb{P}^3$ , necessarily as quartics. It is a quasi-projective variety of dimension 19. For  $m \in \mathcal{F}_4$ , let  $\rho(m)$  denote the Picard number of any surface corresponding to  $m$  and consider the *higher Noether-Lefschetz loci*,

$$\mathrm{NL}_k = \{m \in \mathcal{F}_4 : \rho(m) \geq k\}.$$

For an even non-degenerate lattice  $M$  of signature  $(1, t)$  where  $t \leq 19$ , a K3 surface  $X$  is called  $M$ -polarized if there is a primitive embedding  $M \rightarrow \mathrm{Pic}(X)$ , primitive meaning that the quotient is torsion free. If the image of  $M$  contains a numerically effective element of positive self-intersection, then we call  $X$  pseudo-ample  $M$ -polarized. Generalizing the situation for quasi-polarization, there is a coarse moduli space  $K_M$  of pseudo-ample  $M$ -polarized K3 surfaces. It is a quasi-projective variety that is either empty or of dimension  $20 - \mathrm{rank}(M)$  [11] (see also [20] Proposition 2.3.2 or [22] 12.5 Corollary 4). The exact condition on a lattice  $M$  under which there exists a K3-surface with Picard lattice isomorphic to  $M$  is that  $M$  has signature  $(1, t)$  for some  $1 \leq t \leq 19$  and that  $M$  may be primitively embedded into the K3-lattice  $3U \oplus 2(-E_8)$ , see [22] Corollary 4 or [28] Corollary 1.9. It is clear from the construction in [11] that if  $M$  contains a quasi-polarization of degree 4, then  $K_M$  sits naturally in  $\mathcal{F}_4$  as a closed subset. Moreover, for any  $1 \leq \rho \leq 20$  there exists a smooth quartic surface in  $\mathbb{P}^3$  with Picard number  $\rho$ , see [22] 12.5 Corollary 3 and Corollary 4. Since there are only countably many lattices up to isomorphism we get the following statement.

**Proposition 3.4.** *For  $2 \leq k \leq 20$ ,  $\mathrm{NL}_k$  is a countable union of subvarieties of  $\mathcal{F}_4$  of dimension  $20 - k$ .*

We say that a condition holds for a *very generic* point, or *very general* point, of a variety  $Y$  if there is a countable union  $K = \cup_{i \in \mathbb{Z}} K_i$  of proper closed subsets  $K_i \subset Y$ , such that the condition holds for all points  $y \in Y \setminus K$ .

**Theorem 3.5.** *A very generic Heisenberg invariant quartic  $X \subset \mathbb{P}^3$  has Picard number 16.*

*Proof.* Considering the second claim of Corollary 3.2, it remains to show that  $\rho(X) \leq 16$ . Let  $V \subset U$  be the open subset parameterizing smooth Heisenberg invariant quartics. Then we have a morphism

$$\gamma : V \rightarrow \mathcal{F}_4,$$

such that if  $\gamma(u) = \gamma(v)$ , then  $\mathcal{X}_u$  is isomorphic to  $\mathcal{X}_v$  via an automorphism on  $\mathbb{P}^3$ . Now,  $16 < \rho(\mathcal{X}_v)$  precisely when  $v \in \gamma^{-1}(\mathrm{NL}_{17})$ . Note that  $V$  has dimension 4 and that, by Proposition 3.4,  $\mathrm{NL}_{17}$  is a countable union of 3-folds. Hence it is enough to see that  $\gamma^{-1}(\gamma(v))$  is finite for generic  $v \in V$ . This is the case by upper-semi-continuity of the dimension of  $\gamma^{-1}(\gamma(v))$  and the fact that  $\gamma^{-1}(\gamma(v))$  is finite if  $v$  corresponds to the Fermat quartic, Example 3.3.  $\square$

**Corollary 3.6.** *For very generic  $u \in U$  and  $X = \mathcal{X}_u$ ,  $\mathrm{Pic}(X)^H = \mathbb{Z}h$ , where  $h \in \mathrm{Pic}(X)$  is the class of the hyperplane section. In particular, every invariant curve on  $X$  is a complete intersection between  $X$  and another surface in  $\mathbb{P}^3$ .*

*Proof.* Since  $\text{rank}(\text{Pic}(X)) = 16$ ,

$$\text{rank}(T_X) = \text{rank}(H^2(X, \mathbb{Z})) - \text{rank}(\text{Pic}(X)) = 6.$$

Because  $T_X \cap \text{Pic}(X) = \{0\}$ , and  $T_X$  is contained in the rank 7 lattice  $H^2(X, \mathbb{Z})^H$ , it follows that  $\text{Pic}(X)^H$  has rank at most 1. Clearly, the class  $h$  is invariant under  $H$  and thus  $\text{rank}(\text{Pic}(X)^H) = 1$ . Let  $D$  be a generator of  $\text{Pic}(X)^H$  and suppose that  $h = nD$ , where  $n \in \mathbb{Z}$ . Since  $4 = h^2 = n^2 D^2$ , and  $D^2$  is even since  $\text{Pic}(X)$  is an even lattice, it follows that  $n = \pm 1$ .  $\square$

*Remark 3.7.* The family of Heisenberg invariant quartics gives rise to another 4-parameter family of K3 surfaces by taking the quotients by the Heisenberg group and resolving the resulting singularities. As is outlined below, the very generic member of this family also has Picard number 16. Let  $u \in U$  be general and put  $X = \mathcal{X}_u$ . The quotient  $Q = X/H$  has 15 singularities  $p_1, \dots, p_{15}$  of type  $A_1$  coming from the points of  $X$  with non-trivial stabilizer, see [34] Paragraph 5. Let

$$\pi : Y \rightarrow Q$$

be the minimal resolution. Since the  $H$  action on  $X$  is symplectic,  $Y$  is also a K3 surface [34]. We shall see that  $\rho(Y) = \rho(X)$ . Let  $\alpha : X \rightarrow Q$  be the quotient map and put  $Q' = Q \setminus \{p_1, \dots, p_{15}\}$  and  $X' = \alpha^{-1}(Q')$ . Then  $\alpha : X' \rightarrow Q'$  composed with the inverse  $\pi^{-1} : Q' \rightarrow Y$  is a finite to one morphism. Thus there is a finite to one rational map  $X \rightarrow Y$ , and this implies that  $\rho(X) = \rho(Y)$  [21].

We shall give a few examples of subfamilies of  $\mathcal{X} \rightarrow \mathbb{P}^4$  whose members have higher Picard number.

- (1)  $\rho = 17$ . A general point on the Nieto quintic  $N_5$  corresponds to a smooth quartic  $X$  with 16 disjoint lines  $l_1, \dots, l_{16}$  [4]. Let  $h \in \text{Pic}(X)$  be the hyperplane class. Since the intersection matrix of  $h, l_1, \dots, l_{16}$  has rank 17, these divisors span a rank 17 sublattice of  $\text{Pic}(X)$ . Now,  $X$  is the desingularized Kummer surface of an Abelian surface  $A$  with a  $(1, 3)$ -polarization and a choice of level-2 structure [4]. It is the blowing up of the 16 singular points of the Kummer surface that gives rise to 16 disjoint lines on  $X$ . The moduli space of Abelian surfaces as above is an unbranched double cover of an open subset of  $N_5$ . A general point on  $N_5$  thus corresponds to a general Abelian surface  $A$  with a  $(1, 3)$ -polarization. Then  $\rho(A) = 1$  by [25] 5.3.C. Since  $\rho(X) = \rho(A) + 16$  by [17] Theorem 4.31, we conclude that  $\rho(X) = 17$ .
- (2)  $\rho = 19$ . Consider the pencil  $P$  of surfaces defined by  $x^4 + y^4 + z^4 + w^4 + \lambda xyzw = 0$ ,  $\lambda \in \mathbb{C}$ . A general point on  $P$  corresponds to a smooth surface with Picard number 19 [12]. This property is shared by the 15 pencils arising as  $\mathfrak{S}_6$  translates of  $P$ . The pencil  $P$  has an interesting connection to the family of all quartic surfaces in  $\mathbb{P}^3$  in the context of mirror symmetry, see [11]. The family of all quartics has a 19 dimensional parameter space and the very general Picard number is 1. There is a particular symplectic action of the group  $(\mathbb{Z}/4\mathbb{Z})^2$  on the surfaces in the pencil  $P$  and as in Remark 3.7, taking the quotient and resolving the singularities gives rise to a 1-parameter family of K3 surfaces with general Picard number 19. This 1-parameter family and the 19-parameter family of all quartics are so-called mirror families of each other. Note that the dimension of the parameter space of one family is the very general Picard number of the mirror family.

- (3)  $\rho = 20$ . The Fermat quartic  $X = \{x^4 + y^4 + z^4 + w^4 = 0\}$  is known to have Picard number 20. Let  $p, q \in \mathbb{P}^3$ ,  $p = (1, \alpha, 0, 0)$  and  $q = (0, 0, 1, \beta)$  be such that  $\alpha^4 = \beta^4 = -1$ . The line joining  $p$  and  $q$  is contained in  $X$  and by choosing different combinations of 4<sup>th</sup> roots of  $-1$  and permuting the coordinates, we get 48 lines on  $X$ . One checks that the intersection matrix of these lines has rank 20. The Fermat quartic and its 15 translates by the  $\mathfrak{S}_6$  action thus give examples of smooth surfaces in the family with maximal possible Picard number.

#### 4. CONICS ON THE INVARIANT SURFACES

**Definition 4.1.** Let  $X$  be a quartic surface in  $\mathbb{P}^3$ . We say that a plane  $L$  in  $\mathbb{P}^3$  is a *trope* of  $X$  if  $X \cap L$  is an irreducible conic counted with multiplicity two.

**Lemma 4.2.** *A quartic surface  $X \subset \mathbb{P}^3$  which has a trope  $L$  is necessarily singular.*

*Proof.* Let  $(x, y, z, w)$  be coordinates on  $\mathbb{P}^3$ . Change coordinates so that  $L = \{x = 0\}$ . Then  $X$  is defined by  $x F(x, y, z, w) + (G(y, z, w))^2 = 0$ , for some cubic polynomial  $F$  and quadratic polynomial  $G$ . Then the set  $M = \{x = 0\} \cap \{F = 0\} \cap \{G = 0\}$  is nonempty and inside the singular locus of  $X$ .  $\square$

We shall argue that, for generic  $u \in S_3$ , the Kummer surface  $\mathcal{X}_u$  is uniquely determined by any of its nodes as well as any of its tropes. Let  $F_\lambda(x, y, z, w)$  be a Heisenberg invariant polynomial depending on the parameters  $\lambda = (A, B, C, D, E)$  as in (1). For a general  $p \in \mathbb{P}^3$  there is a unique  $u \in U$  such that  $\mathcal{X}_u$  is singular at  $p$ . To see this, it is enough to check that there is a point  $p = (x, y, z, w) \in \mathbb{P}^3$  such that the system of linear equations

$$\frac{\partial F_\lambda}{\partial x}(p) = \frac{\partial F_\lambda}{\partial y}(p) = \frac{\partial F_\lambda}{\partial z}(p) = \frac{\partial F_\lambda}{\partial w}(p) = 0$$

has a unique solution  $(A, B, C, D, E) \in \mathbb{P}^4$  (this is true for example if  $p = (1, 2, 3, 4)$ ). In fact, there exists a unique invariant quartic singular at  $p \in \mathbb{P}^3$  exactly when  $p$  is not on a fix line [4]. Using  $\mathfrak{S}_6$ -invariance one checks that the set of  $u \in U$  such that  $\mathcal{X}_u$  has a singular point that lies on some fix line is equal to the union of the 15 tangent spaces  $T_t S_3$  where  $\mathcal{X}_t$  is an invariant tetrahedron. We conclude that a generic Heisenberg invariant Kummer surface is determined by any of its nodes. The corresponding statement for tropes is clear once we see that  $p \in \mathbb{P}^3$  is a node precisely when the plane  $p^* \in \mathbb{P}^{3*}$  with the same coordinates as  $p$  is a trope. Let  $p$  be a node, and note that we may assume  $p$  to be a generic point in  $\mathbb{P}^3$ . As in [19] Chapter I Â§3, there are 6 points in the orbit of  $p$  that lie in  $p^*$ . It follows that the 6 points are multiple points of the curve of intersection between  $p^*$  and  $\mathcal{X}_u$ . This curve must then either contain a line as an irreducible component or be a conic of multiplicity two. The first case is excluded by Proposition 2.3. The set of 16 tropes is hence the orbit of  $p^*$ .

**Theorem 4.3.** *A generic invariant quartic  $\mathcal{X}_u$  contains at least 320 smooth conics.*

*Proof.* Pick  $u \in U$  generic and let  $q$  be a node of the Segre cubic  $S_3$ . The line through  $u$  and  $q$  intersects  $S_3$  in one additional point  $p$ , let  $K(p) = \mathcal{X}_p$ . Since  $u$  is generic,  $K(p)$  is a Kummer surface by Proposition 2.2, and thus has 16 tropes which form an orbit under  $H$ . Let  $T$  be a trope of  $K(p)$  and note that  $\mathcal{X}_q$  is a quadric of multiplicity two. The polynomial defining  $\mathcal{X}_u \cap T$ , in homogeneous coordinates

on  $T$ , is a linear combination of squares, and thus reducible. The generic member of the family does not contain any line (Proposition 2.3), and since  $\mathcal{X}_u$  is smooth nor does it have a trope (Lemma 4.2). We conclude that  $\mathcal{X}_u \cap T$  is a union of two smooth conics.

For two different nodes  $q_1$  and  $q_2$  of  $S_3$  the corresponding Kummer surfaces  $K(q_1)$  and  $K(q_2)$  are different, since we may assume that  $u$  is not on the line through  $q_1$  and  $q_2$ . Since a generic Heisenberg invariant Kummer surface is determined by any of its tropes, all tropes of the Kummer surfaces  $K(q)$ , for all of the 10 nodes  $q$  of  $S_3$ , are different. Since two different planes cannot have a smooth conic in common, we conclude that there are at least  $2 \cdot 16 \cdot 10 = 320$  smooth conics on  $\mathcal{X}_u$ .  $\square$

For a general invariant quartic  $X$ , we will refer to the conics in Theorem 4.3 as *the 320 conics* on  $X$ .

## 5. INVARIANT SURFACES CONTAINING LINES

The generic Heisenberg invariant quartic does not contain any line. The invariant surfaces that contain a line are parameterized by the Nieto quintic  $N_5$  and the ten tangent cones to the isolated singularities of  $N_5$  [4]. In this section we will look at the configuration of lines on a Heisenberg invariant quartic that corresponds to a general point on  $N_5$ . This will be used to compute the intersection matrix of the 320 smooth conics on a very generic invariant quartic. General references for this section are [2, 4, 6].

Let  $(p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23})$  be the Plücker coordinates on  $\mathbb{P}^5$ . The Plücker embedding identifies the Grassmannian of lines in  $\mathbb{P}^3$  with the quadric in  $\mathbb{P}^5$  defined by the Plücker relation

$$p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0.$$

We will use so-called Klein coordinates  $(x_0, \dots, x_5)$ , which are defined in terms of Plücker coordinates by

$$\begin{aligned} x_0 &= p_{01} - p_{23}, & x_2 &= p_{02} + p_{13}, & x_4 &= p_{03} - p_{12}, \\ x_1 &= i(p_{01} + p_{23}), & x_3 &= i(p_{02} - p_{13}), & x_5 &= i(p_{03} + p_{12}), \end{aligned}$$

where  $i^2 = -1$ . In Klein coordinates the Plücker relation reads

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0,$$

and the condition that a line with coordinates  $(x_0, \dots, x_5)$  is coplanar with a line with coordinates  $(y_0, \dots, y_5)$  is

$$(2) \quad x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 + x_5y_5 = 0.$$

The Heisenberg group acts on the space of lines in  $\mathbb{P}^3$  and in Klein coordinates the action is given neatly by sign changes in accordance with the table

|            | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|------------|-------|-------|-------|-------|-------|-------|
| $\sigma_1$ | −     | +     | −     | −     | +     | −     |
| $\sigma_2$ | −     | −     | +     | −     | −     | +     |
| $\tau_1$   | +     | +     | −     | −     | −     | −     |
| $\tau_2$   | −     | −     | +     | +     | −     | −     |

where  $\sigma_1, \sigma_2, \tau_1, \tau_2$  are the generators from Section 2.2.

In [4] it is shown that for a generic  $u \in N_5$ ,  $\mathcal{X}_u$  contains exactly 32 lines. In the same paper it is proved that such a surface is a desingularized Kummer surface coming from an Abelian surface  $A$  with a  $(1, 3)$ -polarization. We may assume that

$A$  has Picard number 1 since an Abelian surface which is generic among those admitting a  $(1, 3)$ -polarization has Picard number 1. The Klein coordinates for any of the 32 lines satisfy the condition

$$\frac{1}{x_0^2} + \frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2} + \frac{1}{x_4^2} + \frac{1}{x_5^2} = 0,$$

by which we mean that

$$x_1^2 x_2^2 x_3^2 x_4^2 x_5^2 + x_0^2 x_2^2 x_3^2 x_4^2 x_5^2 + \cdots + x_0^2 x_1^2 x_2^2 x_3^2 x_4^2 = 0.$$

The 32 lines constitute two orbits under the Heisenberg group, each containing 16 lines. There is an involution relating the two orbits: if  $(x_0, \dots, x_5)$  are Klein coordinates for a line in one of the orbits, then

$$\left(-\frac{1}{x_0}, \frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}, \frac{1}{x_4}, \frac{1}{x_5}\right)$$

are Klein coordinates for a line in the other orbit. The sixteen lines in any of the two orbits are mutually disjoint, in fact one of the orbits is the union of the exceptional divisors coming from blowing up the singular Kummer surface in its 16 nodes. The configuration of lines is called a  $32_{10}$  since any line in one orbit intersects exactly 10 lines in the other orbit. There are thus exactly 160 reducible conics on the surface. If two lines, one from each orbit, are coplanar then their *complement* on the surface is an irreducible conic. In other words, the plane that contains the reducible conic intersects the surface in the union of two lines and an irreducible conic. This gives rise to 160 irreducible conics on  $\mathcal{X}_u$ .

**Proposition 5.1.** *For a generic  $u \in N_5$  there are no other irreducible conics on  $\mathcal{X}_u$  except for the 160 complements to reducible conics.*

*Proof.* Let  $X = \mathcal{X}_u$ . By above,  $X$  is a desingularized Kummer surface of an Abelian surface  $A$  with Picard number 1. Let  $e_1, \dots, e_{16}$  denote the halfperiods of  $A$ , that is  $e_i$  is a 2-torsion point or the identity element. Let  $\tilde{A}$  denote the blow-up of  $A$  in  $\{e_1, \dots, e_{16}\}$ . We have morphisms

$$A \xleftarrow{\alpha} \tilde{A} \xrightarrow{\gamma} X,$$

where  $\gamma$  is a double cover branched over the 16 lines  $L_1, \dots, L_{16}$  in one of the orbits and  $\alpha$  is the blow-up map [4]. Let  $E_i = \alpha^{-1}(e_i)$ ,  $i = 1, \dots, 16$ . Then  $E_i \cdot E_j = 0$  if  $i \neq j$ ,  $E_i^2 = -1$  and up to reordering  $\gamma(E_i) = L_i$ . Now, as is explained in [4], if  $M$  is the line bundle corresponding to the sheaf  $\mathcal{O}_X(1)$ , then the line bundle  $\gamma^*(M) \otimes \mathcal{O}_{\tilde{A}}(\sum_{i=1}^{16} E_i)$  on  $\tilde{A}$  descends to a line bundle  $L$  on  $A$  which defines a polarization of type  $(2, 6)$ . Further, there is a symmetric line bundle  $\theta$  on  $A$  of type  $(1, 3)$  such that  $\alpha^*(\theta \otimes \theta) = \gamma^*(M) \otimes \mathcal{O}_{\tilde{A}}(\sum_{i=1}^{16} E_i)$ , see [4]. Let  $T$  denote the divisor class in the Néron-Severi group of  $A$  that corresponds to  $\theta$  and let  $H_X$  be the hyperplane class of  $X$ . Then we have that  $T^2 = 2 \cdot d_1 \cdot d_2$  where  $(d_1, d_2) = (1, 3)$  [30]. Thus  $T^2 = 6$ .

Now let  $C \subseteq X$  be an irreducible conic different from the complements of the reducible conics. By the adjunction formula, we have that  $C^2 = -2$ . Let  $F = \alpha_*(\gamma^*(C))$ . Since  $T^2 = 6$  is square free,  $T$  generates the Néron-Severi group of  $A$  and we may write  $F = dT$  for some integer  $d$ . Now let  $m_i = C \cdot L_i = \gamma^*(C) \cdot E_i$  for

$i = 1, \dots, 16$ . Then  $\alpha^*(F) = \gamma^*(C) + \sum_{i=1}^{16} m_i E_i$ . Hence

$$6d^2 = (dT)^2 = F^2 = \alpha^*(F)^2 = (\gamma^*(C) + \sum_{i=1}^{16} m_i E_i)^2,$$

and therefore

$$6d^2 = \gamma^*(C)^2 + 2 \sum_{i=1}^{16} m_i \gamma^*(C) \cdot E_i + \sum_{i=1}^{16} m_i^2 E_i^2 = 2C^2 + 2 \sum_{i=1}^{16} m_i^2 - \sum_{i=1}^{16} m_i^2.$$

In conclusion,

$$(3) \quad 6d^2 = \sum_{i=1}^{16} m_i^2 - 4.$$

Furthermore,

$$12d = 2T \cdot dT = 2T \cdot F = \alpha^*(2T) \cdot \alpha^*(F) = (\gamma^*(H_X) + \sum_{i=1}^{16} E_i) \cdot (\gamma^*(C) + \sum_{i=1}^{16} m_i E_i),$$

and hence

$$12d = \gamma^*(H_X) \cdot \gamma^*(C) + \sum_{i=1}^{16} m_i E_i \cdot \gamma^*(H_X) + \sum_{i=1}^{16} E_i \cdot \gamma^*(C) + \sum_{i=1}^{16} m_i E_i^2,$$

which implies that

$$(4) \quad 12d = 4 + \sum_{i=1}^{16} m_i + \sum_{i=1}^{16} m_i - \sum_{i=1}^{16} m_i = 4 + \sum_{i=1}^{16} m_i.$$

Since  $L_i$  and  $C$  are not coplanar,  $m_i = 0$  or  $m_i = 1$  for all  $i$ . It follows by (4) that  $d = 1$ . Note also that  $m_i = m_i^2$  for all  $i$ . But then (3) gives  $\sum_{i=1}^{16} m_i = 10$  and (4) gives  $\sum_{i=1}^{16} m_i = 8$ , a contradiction.  $\square$

Let  $u \in N_5$  be generic and put  $X = \mathcal{X}_u$ . As long as all the Klein coordinates  $(x_0, \dots, x_5)$  of a line  $l$  are non-zero, the matter of whether some Heisenberg translate of  $l$  intersects some Heisenberg translate of the image of  $l$  under the involution does not depend on  $l$ . From the description above of the action of the Heisenberg group on the Grassmannian of lines in  $\mathbb{P}^3$ , the condition (2), and the involution relating the two sets of 16 lines on  $X$ , it is straightforward to compute the intersection matrix of the 160 reducible conics on  $X$ . Since the 160 irreducible conics on  $X$  are coplanar with reducible conics, the intersection matrix of all 320 conics on  $X$  is readily deducible from the intersection matrix of the 160 reducible conics. We will not do this in detail but we note in passing that if  $C$  and  $D$  are conics on  $X$  such that  $C$  and  $D$  belong to different orbits under the Heisenberg group  $H$ , then

- $C \cdot gC = 0$  for 6 different  $g \in H$ ,  $g \neq 1$ ,
- $C \cdot gC = 2$  for 9 different  $g \in H$ ,  $g \neq 1$ ,
- $C \cdot gD = 0$  for 4 different  $g \in H$ ,
- $C \cdot gD = 1$  for 8 different  $g \in H$ ,
- $C \cdot gD = 2$  for 4 different  $g \in H$ .

We will see in Section 6 that the intersection matrix of the conics on  $X$  is the same as the intersection matrix associated to the 320 smooth conics on a very generic Heisenberg invariant quartic.

*Remark 5.2.* For later purposes we will consider the intersections of a special set of 16 reducible conics on  $X$ . We need to put an order on the set of all reducible conics on  $X$ . We will order the elements of the Heisenberg group as follows. Let  $g \in H$ ,  $g = \sigma_1^i \sigma_2^j \tau_1^k \tau_2^l$  where  $0 \leq i, j, k, l \leq 1$  and  $\sigma_1, \sigma_2, \tau_1$  and  $\tau_2$  are the generators from Section 2.2. Then we order  $H$  by interpreting  $(i, j, k, l)$  as a number in binary form that marks the place of  $g$ . Let  $L$  be a line on  $X$ . The order on  $H$  puts an order on the orbit of  $L$  as well as the orbit of the image of  $L$  under the involution. This induces an order on the set of reducible conics by saying that if line number  $a$  in the orbit of  $L$  intersects line number  $b$  in the other orbit, and likewise for  $a'$  and  $b'$  with  $(a, b) \neq (a', b')$ , then the reducible conic indexed by  $(a, b)$  is prior to the conic indexed by  $(a', b')$  if and only if  $a < a'$ , or  $a = a'$  and  $b < b'$ . With that order of rows and columns, let  $N$  denote the intersection matrix of the reducible conics on  $X$ . Let  $M$  be the submatrix of  $N$  given by picking out the following rows and columns:

$$(4, 7, 21, 27, 36, 50, 75, 81, 88, 110, 114, 128, 131, 138, 141, 154).$$

This particular choice of conics has been made because, as we shall see in Section 7, it defines a sublattice with minimal discriminant (the conics were found using a computer and a random numbers generator). Then

$$M = \begin{pmatrix} -2 & 0 & 2 & 1 & 2 & 2 & 1 & 0 & 0 & 2 & 0 & 1 & 1 & 2 & 1 & 1 \\ 0 & -2 & 1 & 0 & 2 & 1 & 2 & 0 & 0 & 1 & 2 & 2 & 1 & 2 & 1 & 2 \\ 2 & 1 & -2 & 0 & 1 & 1 & 2 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & -2 & 1 & 0 & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 & -2 & 1 & 1 & 2 & 2 & 0 & 1 & 2 & 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 & 1 & -2 & 1 & 2 & 1 & 0 & 2 & 2 & 1 & 1 & 2 & 1 \\ 1 & 2 & 2 & 2 & 1 & 1 & -2 & 1 & 0 & 2 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 & 1 & -2 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 & 2 & 1 & 0 & 0 & -2 & 2 & 1 & 2 & 0 & 1 & 2 & 1 \\ 2 & 1 & 1 & 1 & 0 & 0 & 2 & 1 & 2 & -2 & 2 & 2 & 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & -2 & 0 & 1 & 1 & 2 & 0 \\ 1 & 2 & 0 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 0 & -2 & 2 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 2 & -2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & -2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 & 1 & 0 & 2 & 1 & 2 & 1 & 0 & 1 & -2 & 1 \\ 1 & 2 & 2 & 2 & 0 & 1 & 1 & 2 & 1 & 0 & 0 & 2 & 0 & 1 & 1 & -2 \end{pmatrix},$$

and  $\det(M) = -512$ .

## 6. THE INTERSECTION MATRIX OF THE 320 CONICS

In this section we show that the configuration of conics (reducible or irreducible) on an invariant quartic  $\mathcal{X}_u$  for generic  $u \in N_5$  is the same as the configuration of the 320 irreducible conics on a very general invariant quartic. That is, the intersection matrices of the two collections of curves are the same up to reordering of the curves. The 320 smooth conics vary in a family of curves whose configuration we can compute in a special case, namely the case of a surface that contains lines. The argument is indirect in that we will make use of some general statements on specialization and the behavior of Néron-Severi groups in families [27, 35].

**Lemma 6.1.** *The configuration of 320 irreducible conics on a very general Heisenberg invariant quartic is the same as the configuration of conics (reducible or irreducible) on a Heisenberg invariant quartic that corresponds to a general point on the Nieto quintic.*

*Proof.* Let  $V \subset U \cong \mathbb{P}^4$  be the open set parameterizing smooth Heisenberg invariant quartics and consider the smooth and proper family  $\mathcal{Y} = \pi^{-1}(V) \rightarrow V$  where  $\pi : \mathcal{X} \rightarrow U$  is the projection.



We shall see that, for a very general  $v \in V$ , there is a monomorphism of groups  $\text{Pic}(\mathcal{Y}_v) \hookrightarrow \text{Pic}(\mathcal{Y}_{v'})$  for any  $v' \in V$ . The Néron-Severi group of the geometric generic fiber of  $\mathcal{Y}$  injects into  $\text{Pic}(\mathcal{Y}_w)$  for any  $w \in V$  (see [27] Proposition 3.6 (a)). Moreover, for very general  $w \in V$ , that injection is an isomorphism (see [27] Proposition 3.6 (b) and Corollary 3.10). Hence, for a very general  $v \in V$  and any  $v' \in V$ ,  $\text{Pic}(\mathcal{Y}_v)$  injects into  $\text{Pic}(\mathcal{Y}_{v'})$ . We need two more properties of the injection  $i : \text{Pic}(\mathcal{Y}_v) \hookrightarrow \text{Pic}(\mathcal{Y}_{v'})$ . First of all, it respects intersection numbers (see [35] X, (7.9.3)). Secondly, by [35] X Proposition 7.3, the following diagram commutes

$$\begin{array}{ccc} \text{Pic}(\mathcal{Y}) & \xrightarrow{r_1} & \text{Pic}(\mathcal{Y}_{v'}) , \\ & \searrow r_2 \quad \nearrow i & \\ & \text{Pic}(\mathcal{Y}_v) & \end{array}$$

where  $r_1$  is the restriction map  $j^*$  with  $j : \mathcal{Y}_{v'} \hookrightarrow \mathcal{Y}$  the inclusion and similarly for  $r_2$ .

Now let  $v' \in N_5$  be such that there are only 320 conics on  $\mathcal{Y}_{v'}$ , which is true for a general point  $v' \in N_5$  by Proposition 5.1. Further, let  $v \in V$  be such that there is a monomorphism of lattices

$$i : \text{Pic}(\mathcal{Y}_v) \hookrightarrow \text{Pic}(\mathcal{Y}_{v'})$$

and such that  $\mathcal{Y}_v$  contains the configuration of 320 smooth conics. Put  $X = \mathcal{Y}_v$  and  $Y = \mathcal{Y}_{v'}$ . Let  $C \in \text{Pic}(X)$  be a class corresponding to one of the 320 smooth conics. We want to show that  $i(C)$  is the class of a conic on  $Y$ . Since  $i(C)^2 = -2$ , either  $i(C)$  or  $-i(C)$  is effective, see [3] Proposition 3.6 (i). Consider the polarizations on  $X$  respectively  $Y$  that define the embeddings as quartic surfaces in  $\mathbb{P}^3$  that come with the definition of the family  $\mathcal{X}$ . For a general hyperplane  $L \subset \mathbb{P}^3$ , these polarizations on  $X$  and  $Y$  both come from the divisor class  $(\mathbb{P}^4 \times L) \cap \mathcal{Y}$  on  $\mathcal{Y}$  via the restriction maps. Hence,  $i$  maps the polarization of  $X$  to the polarization of  $Y$ . It follows that the degree of  $i(C)$  is 2. We conclude that  $i(C)$  is effective. Since  $i(C)^2 = -2$ ,  $i(C)$  is represented either by an irreducible conic or a reducible conic. But there are only 320 conics on  $Y$ , and hence their configuration must be the same as the configuration of the 320 smooth conics on  $X$ .  $\square$

## 7. THE PICARD GROUP

In this section we determine the Picard group of a very general Heisenberg invariant quartic and show that it is generated by the 320 conics on the surface.

It has been known since a hundred years that every curve  $C$  on a general Kummer surface  $Y$  is such that  $C$  counted with multiplicity 2 is a complete intersection between  $Y$  and some other surface in  $\mathbb{P}^3$  [19] Chapter XIII. A modern treatment is given in [16, 17]. In particular, every curve on  $Y$  has even degree. The Kummer surfaces that appear in the family  $\mathcal{X}$  are Heisenberg invariant but this merely reflects a choice of coordinates on  $\mathbb{P}^3$ ; every Kummer surface in  $\mathbb{P}^3$  (coming from an irreducible principally polarized Abelian surface) is projectively equivalent to a Heisenberg invariant Kummer surface, see [10] Theorem 10.3.14 or [17].

**Lemma 7.1.** *For a very generic Heisenberg invariant quartic surface  $X$ , no curve on  $X$  has odd degree.*

*Proof.* Let  $\text{Hilb}_{(d,g)}(\mathbb{P}^3)$  denote the Hilbert scheme of curves in  $\mathbb{P}^3$  with Hilbert polynomial  $P(x) = dx + (1 - g)$ . Recall that  $\text{Hilb}_{(d,g)}(\mathbb{P}^3)$  is projective, see [24] 1.4. Let  $\mathcal{Y} \rightarrow \text{Hilb}_{(d,g)}(\mathbb{P}^3)$  be the universal family. Define an incidence  $I_{(d,g)}$  by

$$\text{Hilb}_{(d,g)}(\mathbb{P}^3) \times U \supseteq I_{(d,g)} = \{(s, u) : \mathcal{Y}_s \subset \mathcal{X}_u\},$$

and let  $\pi : \text{Hilb}_{(d,g)}(\mathbb{P}^3) \times U \rightarrow U$  denote the projection. Now, since for odd  $d$  and any  $g$ , the parameter point of a general Heisenberg invariant Kummer surface is not in  $\pi(I_{(d,g)})$ , we have that  $\pi(I_{(d,g)}) \neq U$  if  $d$  is odd. A surface whose parameter point is outside the countable union  $\bigcup_{k,g} \pi(I_{(2k+1,g)})$  has no curve of odd degree.  $\square$

Finite Abelian groups with a symplectic action on some K3 surface have been classified by Nikulin [34]. In that paper, Theorem 4.7, it is shown that if  $G$  is finite Abelian and acts symplectically on a K3 surface  $X$ , then the induced action of  $G$  on  $H^2(X, \mathbb{Z})$  is independent of the surface, up to isomorphism of lattices. It follows that the lattices  $H^2(X, \mathbb{Z})^G$  and  $\Omega_G = (H^2(X, \mathbb{Z})^G)^\perp$  are determined by  $G$  up to isomorphism. In [13] these lattices are worked out in each case of the classification and in the case of the Heisenberg group,  $\Omega_H$  turns out to be well known. It is isomorphic to  $-\Lambda_{15}$ , where  $\Lambda_{15}$  is the so-called laminated lattice of rank 15, see [8].

*Remark 7.2.* The discriminant of  $\Lambda_{15}$  is  $2^9$  and  $\Lambda_{15}$  is positive definite [8]. In one basis, the bilinear form on  $\Lambda_{15}$  is given by

$$\begin{pmatrix} 4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ -2 & 4 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -2 & 4 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 2 & 0 & 4 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 4 & 2 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 & 4 & 2 & 2 & 1 & 2 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 4 & 2 & 0 & 2 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & 4 & 0 & 2 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 4 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 4 & 2 & 2 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 4 & 1 & 2 & 2 \\ 0 & -1 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 1 & 4 & 0 & 2 \\ 1 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & 0 & 1 & 1 & 2 & 0 & 4 & 2 \\ 1 & 0 & 1 & 1 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 2 & 2 & 4 \end{pmatrix}.$$

**Theorem 7.3.** *Let  $X$  be a very generic Heisenberg invariant quartic and let  $h \in \text{Pic}(X)$  be the hyperplane class. Then*

- (1) *the sublattice  $\mathbb{Z}h \oplus \Omega_H \subset \text{Pic}(X)$  has index 2,*
- (2)  *$\text{discr}(\text{Pic}(X)) = -2^9$ .*

*Proof.* We first show that the group  $\text{Pic}(X)/(\mathbb{Z}h \oplus \Omega_H)$  is cyclic. Since  $\text{Pic}(X)^H = \mathbb{Z}h$  and  $H^2(X, \mathbb{Z})^H$  has rank 7,  $\mathbb{Z}h \oplus T_X \subseteq H^2(X, \mathbb{Z})^H$  is a full rank sublattice. If  $l \in \text{Pic}(X)$  and  $lh = 0$ , then  $l$  kills  $\mathbb{Z}h \oplus T_X$  and therefore  $l \in \Omega_H$ . Since  $\Omega_H \subset \text{Pic}(X)$ , it follows that  $\Omega_H$  is the orthogonal complement of  $\mathbb{Z}h$  in  $\text{Pic}(X)$ . Observe that  $\Omega_H$  is a primitive sublattice of  $\text{Pic}(X)$ , that is if  $nl \in \Omega_H$  for some  $n \in \mathbb{Z}$  and  $l \in \text{Pic}(X)$ , then  $l \in \Omega_H$ . In other words  $\text{Pic}(X)/\Omega_H$  is torsion free. Because  $\text{Pic}(X)/\Omega_H$  has rank 1, there is an element  $l \in \text{Pic}(X)$  such that  $\text{Pic}(X) = \mathbb{Z}l \oplus \Omega_H$ , as Abelian groups. The class of  $l$  in  $\text{Pic}(X)/(\mathbb{Z}h \oplus \Omega_H)$  generates  $\text{Pic}(X)/(\mathbb{Z}h \oplus \Omega_H)$ . Further,

note that for  $v \in (\mathbb{Z}h \oplus \Omega_H)$ ,  $v = nh + \omega$  with  $\omega \in \Omega_H$  and  $n \in \mathbb{Z}$ , we have that  $vh = nh^2 = 4n$ . If it were true that  $\text{Pic}(X) = \mathbb{Z}h \oplus \Omega_H$  then the degree of any curve on  $X$  would be a multiple of 4, but this is not the case since  $X$  contains a conic. Now let  $D \in \text{Pic}(X)$ . It follows by Lemma 7.1 that  $Dh$  is even, say  $Dh = 2m$ . Then  $(2D - mh)h = 0$  and hence  $(2D - mh) \in \Omega_H$ . Consequently,  $2D \in \mathbb{Z}h \oplus \Omega_H$  and thus  $\mathbb{Z}h \oplus \Omega_H$  has index 2 in  $\text{Pic}(X)$ .

Since

$$\text{discr}(\mathbb{Z}h \oplus \Omega_H) = 4 \cdot \text{discr}(\Omega_H) = -2^{11}$$

and

$$\text{discr}(\mathbb{Z}h \oplus \Omega_H) / \text{discr}(\text{Pic}(X)) = 2^2,$$

we have that  $\text{discr}(\text{Pic}(X)) = -2^9$ .  $\square$

**Corollary 7.4.** *The Picard group of a very generic Heisenberg invariant quartic  $X$  is generated by the 320 conics.*

*Proof.* By Lemma 6.1, some set of 16 conics on  $X$  correspond to the 16 conics of Remark 5.2. Let  $P \subseteq \text{Pic}(X)$  be the sublattice generated by these 16 conics on  $X$ . Because  $\text{discr}(P) = \text{discr}(\text{Pic}(X))$ , the index of  $P$  in  $\text{Pic}(X)$  is equal to 1, that is  $P = \text{Pic}(X)$ .  $\square$

The proof of the corollary shows that (in some basis) the lattice structure of the Picard group of a very generic Heisenberg invariant quartic is given by the matrix  $M$  in Remark 5.2.

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